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# Multiplying unitary random matrices—universality and spectral properties

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## Abstract

In this paper, we calculate, in the large  $N$  limit, the eigenvalue density of an infinite product of random unitary matrices, each of them generated by a random Hermitian matrix. This is equivalent to solving the unitary diffusion generated by a Hamiltonian random in time. We find that the average eigenvalue density is universal and depends only on the second moment of the generator of the stochastic evolution. We find indications of critical behaviour (eigenvalue spacing scaling like  $1/N^{3/4}$ ) close to  $\theta = \pi$  for a specific critical evolution time  $t_c$ .

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## 1. Introduction

A key feature of the random matrix theory is that many properties of random matrix models do not depend on the fine details of these models but only on some very general symmetry properties and a very limited number of numerical coefficients (usually just a single coefficient is enough). These universal properties facilitate the widespread applications of random matrix models in various fields since one can use the models to learn something about the behaviour of complex systems without knowing all the precise microscopic details of these systems. A wide range of applications of matrix models can be found in a review article [1].

The simplest example of such behaviour is the eigenvalue density of a matrix with entries independently distributed according to some probability distribution. Then the eigenvalue density in the  $N \rightarrow \infty$  limit follows Wigner's semicircle law with the scale set just by the second moment of the distribution. All the dependence on other properties of the initial probability distribution disappears. In this work, we will uncover a similar type of universality in a different context.

The aim of this paper is to study the properties of products of unitary random matrices in the planar (large  $N$ ) limit and focus on the case when the number of factors in the product becomes infinite. This can be interpreted as a multiplicative diffusion process on the unitary group<sup>1</sup>. A natural physical interpretation would be the quantum-mechanical evolution governed by a Hamiltonian which changes randomly in time. Another possibility would be the modelling of Wilson loops in the lattice gauge theory. In this paper, we will not further examine these possible applications but rather concentrate on mathematically solving the model.

We show that the eigenvalue density exhibits universality properties, i.e. it only depends on the second moment of the random Hamiltonian which generates the stochastic evolution. But of course the resulting eigenvalue density is much more complex than the semicircle law. We derive equations for the eigenvalue density, give an explicit expression for the lowest moments and study the properties of the model close to a phase transition where the eigenvalue support begins to cover the whole unit circle.

## 2. Multiplicative unitary diffusion

We consider the product of  $M$  unitary  $N \times N$  random matrices  $U_k$  in the  $M, N \rightarrow \infty$  limit:

$$U = \lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \prod_{k=1}^M U_k \quad (1)$$

where the  $U_k$  are generated by

$$U_k = e^{i\varepsilon H_k} \quad (2)$$

and where  $\varepsilon = \sqrt{t/M}$ . Such a scaling is standard for diffusive processes and also works very well for matrix-valued diffusion processes studied in [2, 3].  $t$  is then a real parameter corresponding to a ‘diffusive’ evolution time and the continuum limit  $M \rightarrow \infty$  exists. The generators of the evolution  $H_k$  are  $N \times N$  Hermitian matrices drawn from a probability distribution

$$P(H) \sim e^{-N \text{tr} V(H)} \quad (3)$$

where we assume that the first moment  $m_1 = \langle \frac{1}{N} \text{tr} H \rangle$  vanishes,  $m_1 = 0$ . We will show below that the spectral properties of (1) depend only on the second moment  $m_2$  of the distribution (3)

$$m_2 = \left\langle \frac{1}{N} \text{tr} H^2 \right\rangle. \quad (4)$$

The main aim of this paper is to find the eigenvalue distribution of the product (1)

$$\rho(\theta, t) = \left\langle \frac{1}{N} \sum_{j=1}^N \delta(\theta - \theta_j) \right\rangle, \quad (5)$$

where  $\theta_j$  are the phases of the eigenvalues  $e^{i\theta_j}$  of  $U$  defined through (1).

In the next section, we will use free random variable methods to derive an equation from which one can get  $\rho(\theta, t)$ . Sometimes we will omit the second argument but of course the dependence on  $t$  will be there.

<sup>1</sup> Recently, matrix-valued multiplicative diffusion has been considered for  $2 \times 2$  real matrices in [2] and for infinite Hermitian and complex matrices in [3].

### 3. S-transform method

The main difficulty encountered in [3], when considering products of random matrices, was the necessity to deal with non-Hermitian matrices and eigenvalues covering two-dimensional regions of the complex plane. Here fortunately, since the product matrices are always unitary, the eigenvalues lie on the unit circle and hence the eigenvalue density can be uniquely reconstructed from just the knowledge of the moments, which are encoded in the asymptotic expansion of the Green function<sup>2</sup>

$$G(z) = \int_0^{2\pi} \frac{\rho(\theta)}{z - e^{i\theta}} d\theta. \tag{6}$$

Our aim now is to obtain the spectral density  $\rho_{\text{PROD}}(\theta)$  of the product  $U = \prod_{k=1}^M U_k$ , equivalently the corresponding Green function  $G_{\text{PROD}}(z)$ , from the spectral density  $\rho_H(\theta)$  for the generator  $H$ .

To do that we use  $S$ -transforms introduced in [4]. Firstly, we define an auxiliary function  $\chi(z)$  through

$$\frac{1}{\chi} G\left(\frac{1}{\chi}\right) - 1 = z. \tag{7}$$

Then the  $S$ -transform is

$$S(z) = \frac{1+z}{z} \chi(z). \tag{8}$$

Their main property is that the  $S$ -transform of a product of random matrix ensembles is a product of  $S$ -transforms of the individual factors [4]. Note that since  $S(z)$  can be obtained from the Green function  $G(z)$ , it is defined (as is  $G(z)$ ) *after* ensemble averaging. Before we apply this setup to (1), let us note that putting together the two previous equations we arrive at a functional relation satisfied by  $S$  and  $G$ :

$$\frac{1}{zS} G\left(\frac{1+z}{z} \frac{1}{S}\right) = 1. \tag{9}$$

In our case, all single matrix Green functions are the same since they come from the same distribution, so we can write

$$S_{\text{PROD}}(z) = \lim_{M \rightarrow \infty} \prod_{i=1}^M S_i(z) = \lim_{M \rightarrow \infty} (S_1(z, \varepsilon))^M. \tag{10}$$

Let us now find  $S_1(z, \varepsilon)$ . The phases of the eigenvalues of  $U_1 = e^{i\varepsilon H}$  are distributed with the density<sup>3</sup>

$$\rho_1(\theta, \varepsilon) = \frac{1}{\varepsilon} \rho_H\left(\frac{\theta}{\varepsilon}\right) \tag{11}$$

where  $\varepsilon = \sqrt{t/M}$ . Since we will be interested in taking the limit  $M \rightarrow \infty$  we may perform an expansion in  $\varepsilon$ . Inserting (11) and the definition (6) into (9) leads to an equation for  $S_1(z, \varepsilon)$ :

$$\int \frac{1}{\varepsilon} \frac{\rho_H\left(\frac{\theta}{\varepsilon}\right)}{1 + z - e^{i\theta} z S_1(z, \varepsilon)} d\theta = 1. \tag{12}$$

<sup>2</sup> For an explicit formula see equations (16) and (30).

<sup>3</sup>  $\rho_H(x) \equiv 0$  outside the eigenvalue support of  $H$ .

From the form of (10), we see that we need to calculate  $S_1$  only to the order  $\mathcal{O}(\varepsilon^2) \sim \mathcal{O}(1/M)$ . Substituting  $u = \theta/\varepsilon$  and expanding in the Taylor series in  $\varepsilon$  we obtain

$$\int \rho_H(u) \left( 1 + i\varepsilon uz + \left( -u^2 z^2 - \frac{1}{2} u^2 z + sz \right) \varepsilon^2 + \mathcal{O}(\varepsilon^3) \right) du = 1 \quad (13)$$

where  $s = s(z)$  comes from the Taylor expansion<sup>4</sup>  $S_1(z, \varepsilon) = 1 + s(z)\varepsilon^2 + \mathcal{O}(\varepsilon^3)$ .

We may now calculate  $s(z)$

$$s(z) = (z + 1/2)\langle u^2 \rangle \equiv (z + 1/2)m_2. \quad (14)$$

From (10), we may now obtain the  $S$ -transform for the product

$$\begin{aligned} S_{\text{PROD}} &= \lim_{M \rightarrow \infty} (S_1(z, \varepsilon))^M \Big|_{\varepsilon = \sqrt{\frac{t}{M}}} \\ &= \lim_{M \rightarrow \infty} \left( 1 + \frac{t}{M} s(z) + \mathcal{O} \left( \left( \frac{t}{M} \right)^{3/2} \right) \right)^M = e^{t(z + \frac{1}{2})m_2}. \end{aligned} \quad (15)$$

This result shows that  $S_{\text{PROD}}$  depends *only* on the second moment  $m_2$  of  $H$ . So in the limit  $M \rightarrow \infty$ , we obtain universal behaviour of the system independent of the spectral density of the generator of the stochastic evolution  $H$ , as long as the first moment vanishes (no drift) and the second moment is finite. It will be interesting to consider the cases where these assumptions are violated which would lead to anomalous diffusion. We leave these problems for future investigation.

The final step is to come back from  $S_{\text{PROD}}$  to the Green function  $G_{\text{PROD}}(z, t)$  (from now on we drop the subscript). It is convenient to introduce an auxiliary function  $f(z, t)$  as

$$G(z, t) = \frac{1 + f(t, z)}{z}. \quad (16)$$

It is easy to check that  $f$  fulfils an equation:  $f\left(\frac{1}{\chi(z)}\right) = z$ . This means that  $f$  and  $1/\chi$  are functional inverses of each other, so the following relation  $1/\chi(f) = z$  is also true. This observation, together with the result (15) and the definition (8), leads us to the final equation

$$zf = (1 + f) e^{-t(f + \frac{1}{2})m_2}. \quad (17)$$

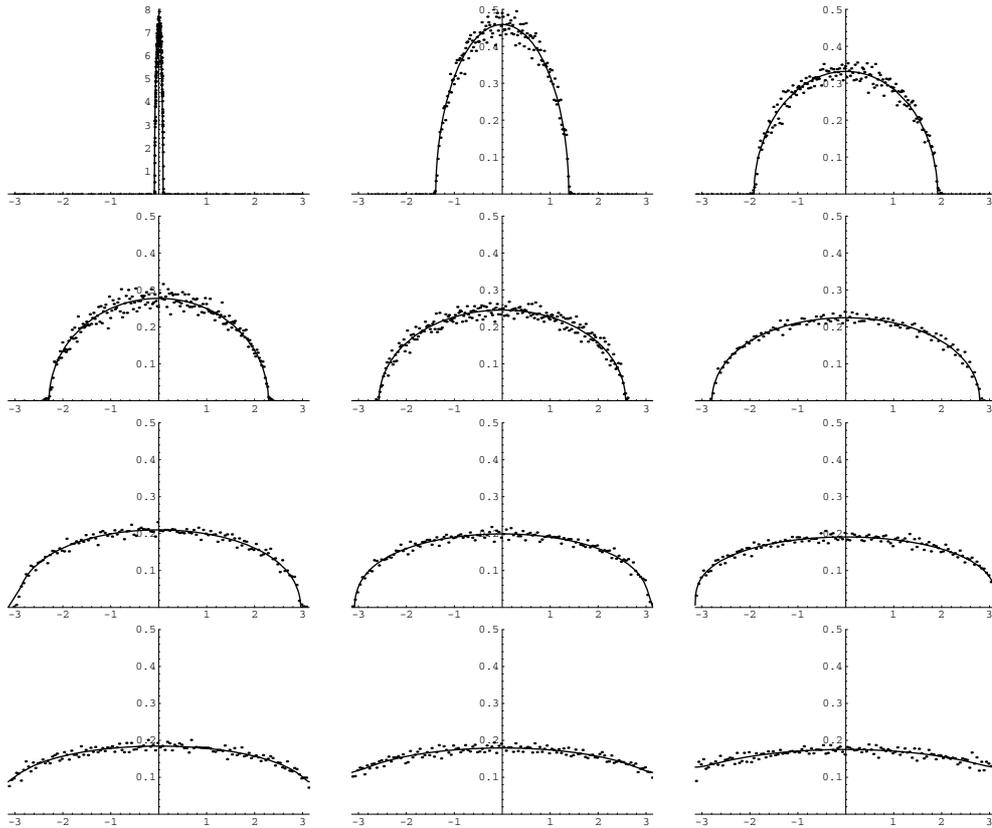
This equation encodes all the spectral properties of the unitary diffusion process (1). In the next section, we will proceed to investigate some of its properties.

#### 4. The dynamical properties of the unitary diffusion

In this section, we analyse the dynamical behaviour of the unitary matrix diffusion. For small times  $t$ , the eigenvalues will be concentrated only in a small neighbourhood of  $\theta = 0$ . For longer times, the support of the eigenvalue density  $\rho$  will expand and when some critical time  $t_c$  is reached the eigenvalues will fill the whole circle. But of course the eigenvalue density  $\rho$  will be nonuniform. In fact, we expect critical behaviour close to  $\theta = \pi$  with a nonstandard fractional eigenvalue spacing. Only later for  $t \rightarrow \infty$  will the eigenvalues become uniformly spread over the whole unit circle. In figure 1, we show numerical results for the eigenvalue density obtained by generating unitary matrices and compare it to the one extracted from (17) (see below).

In this section, we will quantitatively analyse this behaviour.

<sup>4</sup> Here, we used the assumption that the first moment of  $H$  vanishes.



**Figure 1.** The time evolution of the spectral function  $\rho(\theta, t)$  (numerical simulation). The following pictures show  $\rho(\theta, t)$  after a time  $t = 1, 250, 500$  up to 2750 (we took  $m_2 = 1/500$ ). The ninth graph shows the behaviour at the critical time  $t = t_c$  when the gap in the spectrum vanishes. In the simulations, we used products of 60 matrices, each of size  $80 \times 80$ .

4.1. The support of the eigenvalue distribution and the critical time  $t_c$

Although one cannot find an analytical formula for the eigenvalue density, one can analytically find the edges of the eigenvalue support. These occur when the Green function has an infinite derivative  $\partial_z G = \infty$ . Differentiating (17) with respect to  $z$  gives

$$1 = -\frac{\partial_z f}{f^2} (1 + m_2 t f + m_2 t f^2) e^{-t(f + \frac{1}{2})m_2}. \tag{18}$$

So, the end points are determined through the solutions of the equation

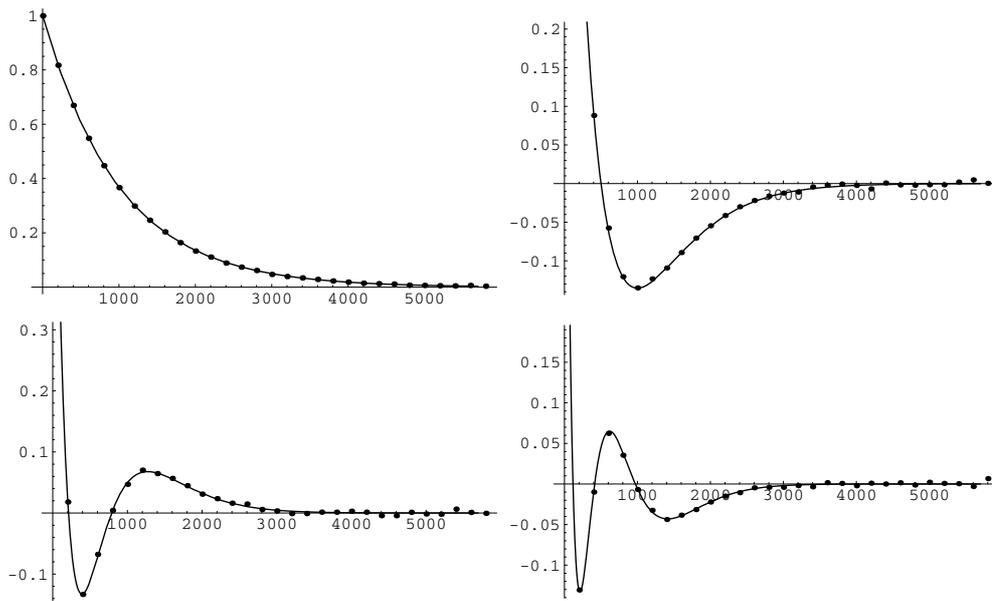
$$1 + m_2 t f + m_2 t f^2 = 0. \tag{19}$$

Once we know  $f$ , we can reconstruct the end points  $z$  using (17). The result is

$$z_{\text{edge}} = \frac{\sqrt{4 - m_2 t} + i\sqrt{m_2 t}}{\sqrt{4 - m_2 t} - i\sqrt{m_2 t}} e^{\frac{i}{2}\sqrt{m_2 t}\sqrt{4 - m_2 t}} \tag{20}$$

and its complex conjugate  $z_{\text{edge}}^*$ . When these two solutions are equal ( $z_{\text{edge}} = z_{\text{edge}}^* = -1$ ), the eigenvalues will cover the whole circle. This will happen for the critical time

$$t_c = \frac{4}{m_2}. \tag{21}$$



**Figure 2.** The comparison of numerical simulations (dots) with analytical results (lines), for  $n = 1, 2, 3, 4$  and  $m_2 = 1/500$ .

#### 4.2. The moments of $U$

Another quantity which can be analytically calculated is the moments of  $U$ . The coefficients of the auxiliary function  $f$  around  $z = \infty$

$$f(z) = \sum_{k=1}^{\infty} \frac{a_k}{z^k} \quad (22)$$

are indeed directly linked to the moments:

$$a_k = \left\langle \frac{1}{N} \text{tr } U^k \right\rangle. \quad (23)$$

So, inserting (22) into (17) allows us to find the moments iteratively. The expressions for the lowest ones are

$$a_1 = e^{-\frac{1}{2}m_2 t} \quad (24)$$

$$a_2 = -e^{-m_2 t} (-1 + m_2 t) \quad (25)$$

$$a_3 = \frac{1}{2} e^{-\frac{3m_2 t}{2}} (2 - 6m_2 t + 3m_2^2 t^2) \quad (26)$$

$$a_4 = -\frac{1}{3} e^{-2m_2 t} (-3 + 18m_2 t - 24m_2^2 t^2 + 8m_2^3 t^3) \quad (27)$$

$$a_5 = \frac{1}{24} e^{-\frac{5m_2 t}{2}} (24 - 240m_2 t + 600m_2^2 t^2 - 500m_2^3 t^3 + 125m_2^4 t^4). \quad (28)$$

In figure 2, we compare the formulae for the lowest four moments with numerical simulations of the unitary matrix diffusion and find complete agreement.

4.3. The eigenvalue density

Equation (17) allows us to directly reconstruct the Green function. However, it is also very simple to recover the eigenvalue density. This follows from the observation that the moments of  $U$  are just the Fourier coefficients of the eigenvalue density  $\rho(\theta)$

$$\left\langle \frac{1}{N} \text{tr} U^k \right\rangle = \int_0^{2\pi} \rho(\theta) e^{ik\theta}. \tag{29}$$

Using the relation of  $f$  to moments derived earlier, and the symmetry  $\rho(\theta) = \rho(-\theta)$  one finally finds

$$\rho(\theta) = -\frac{1}{2\pi} \text{Re} \left( \frac{1}{2} + f \right). \tag{30}$$

Equation (17) may be easily solved numerically. In figure 1, we show the resulting eigenvalue density together with numerical simulations for various times  $t$ .

4.4. Critical behaviour at  $t = t_c$  and level spacing

At  $t = t_c$ , the edges of the eigenvalue support touch at  $z = -1$ . Typically, in such cases we expect a new critical type of behaviour and nonstandard scaling of the eigenvalue spacing with  $N$ . Let us now analyse this behaviour. Inserting  $t = t_c$  and  $f = -1/2 + F$  into (17) we obtain

$$z = \frac{F + 1/2}{F - 1/2} e^{-4F}. \tag{31}$$

To find the behaviour close to  $z = -1$  (equivalent to  $\theta = \pi$ ), we expand the left-hand side of (31) in  $F$  and put  $z = -1 + iy$  to get

$$-1 + iy \approx -1 - \frac{16F^3}{3}. \tag{32}$$

Using the relation between  $f$  and the eigenvalue density (30), we thus find the behaviour close to  $\theta = \pi$ :

$$\rho(\theta) \sim \left\{ \frac{1}{2\pi} \left( \frac{3}{16} \right)^{\frac{1}{3}} \cos \frac{\pi}{6} \right\} \cdot |\theta - \pi|^{\frac{1}{3}}. \tag{33}$$

Such behaviour of the eigenvalue density leads to nonstandard eigenvalue spacing and signifies the appearance of a new universal regime on the scale of eigenvalue spacing (analogous to Airy universality and  $1/N^{2/3}$  spacing on the edges of the eigenvalue distribution of a generic Hermitian random matrix [5–8] in contrast to the standard  $1/N$  spacing in the classical Wigner–Dyson regime [9]).

In our case, the number of eigenvalues between  $\pi$  and  $\Lambda$  is approximately equal to  $n \sim N(\Lambda - \pi)^{4/3}$ . Re-expressing  $\Lambda$  in terms of  $n$  shows that the eigenvalue spacing in the vicinity of  $\theta = \pi$  scales like  $1/N^{3/4}$ . A similar scaling appeared in a deterministic + random Hermitian random matrix model [10] as well as in a certain class of chiral random matrix models at finite temperature [11]. It would be interesting to compare these regimes and/or try to apply the methods of [12] to the case at hand. We leave this problem for further investigation.

We can also investigate the behaviour for times smaller than  $t_c$ . We can repeat our calculations for the time  $t_\varepsilon = \frac{4-\varepsilon^2}{m_2} < t_c$ . We can calculate the corresponding  $f_\varepsilon$  at the edge of the spectrum from equation (19) with  $t = t_\varepsilon$ . After that we substitute  $f = f_\varepsilon + F$  and obtain  $z$  from (17). After expanding in the Taylor series with respect to  $\varepsilon$  and  $F$ , we get the result

$$z = -1 + 4i\varepsilon F^2 - \frac{16F^3}{3}. \tag{34}$$

We also expand  $z_{\text{edge}}$  for  $t = t_\varepsilon$  in  $\varepsilon$  and finally obtain

$$z - z_{\text{edge}} = 4i\varepsilon F^2 - \frac{16F^3}{3}. \quad (35)$$

For  $\varepsilon \neq 0$  ( $t < t_c$ ), the edge of the spectrum behaves like the edge of the standard Wigner semicircle, i.e. like  $x^{1/2}$ . Only in the limit  $\varepsilon \rightarrow 0$ , which corresponds to  $t = t_c$ , do we obtain critical behaviour like  $x^{1/3}$ .

## 5. Discussion

In this paper, we considered the multiplicative unitary matrix diffusion generated by random Hermitian matrices. We found the eigenvalue density as a function of the evolution time in the large  $N$  limit using  $S$ -transform methods. The eigenvalue distribution turns out to be *universal* and depends only on the second moment of the random Hermitian matrix which generates the diffusion process.

We found that at a critical time of evolution  $t = t_c$ , the eigenvalues start to fill the whole unit circle, and close to  $\theta = \pi$  a nonstandard eigenvalue spacing  $\sim 1/N^{3/4}$  sets in which signifies the appearance of a new critical regime.

There are various further issues that one could investigate. Firstly, relaxing the assumption of the existence of the second moment might lead to defining anomalous diffusion processes. Secondly, it would be interesting to study microscopic properties of these unitary matrices; however, in order to do that new methods have to be developed. Thirdly, a more detailed investigation of the critical behaviour at  $t = t_c$  close to  $\theta = \pi$  would be interesting and last but not the least the application of these results to some physical situations.

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